

# World history of mathematics

## Mathematics in Africa



Mathematical  
Institute

Mathematics has emerged in Africa in many varied forms over the millennia. Different civilisations have employed sophisticated arrangements of weights and measures, elaborate counting systems, and have played mathematical games.

Beyond that of the North African Islamic and ancient Egyptian civilisations, the history of mathematics in Africa has been little-studied, and sources are sketchy. Further research will undoubtedly uncover more of the details of African mathematics that have been hidden for centuries.



▲ The Ishango bone, controversial evidence of early arithmetic in Africa

### Early arithmetic?

The Ishango bone is a tool made from a baboon fibula, discovered in 1960 in what is now the Democratic Republic of the Congo. It dates to between 20,000 and 18,000 BCE. Notches carved into the bone have been interpreted as tally marks, evidence of an arithmetical game, a record of lunar cycles, or simply as grooves to aid grip and therefore entirely non-mathematical.

### Ancient Egypt

The ancient Egyptian civilisation arose around 3,000 BCE and lasted

in various forms for the next 3,000 years. Comprehensive evidence of Egyptian mathematics is rare, since it would have been recorded on papyrus, which has not survived. However, the very few remaining sources allow us to reconstruct the sophisticated arithmetical techniques that Egyptian scribes used in record-keeping. We also have evidence of geometrical understanding: for example, a standard procedure for calculating the area of a circle.

Alexandria in Egypt was also home, in the fourth century CE, to one of the first female mathematicians about whom we have any substantial information: Hypatia, who wrote commentaries on ancient mathematical texts, and is said to have constructed astronomical instruments.



▲ Hypatia of Alexandria (?–415 CE), as depicted by Jules Maurice Gaspard in 1908

### Weights in West Africa

The small hard seeds of the rosary pea were traditionally used as standard weights by various peoples in West Africa, owing to their consistency in size. However, metal weights became the norm following the introduction of metal-working techniques to present-day Ghana, and the expansion of the trans-Saharan

▼ Akan gold-weights, and a balance (18th–20th century CE); the weight on the left depicts a drummer, although the arms have been removed – possibly to adjust the weight



gold trade, in the 14th century CE. The Akan people in particular produced brass weights, for weighing gold, that were both elaborately and accurately crafted.

### Mathematical Games

Games of chance and strategy have been played throughout Africa over the centuries. One example is *morabaraba*, a version of the Roman game of nine men's morris, played in South Africa and Botswana. Another game that is particularly widespread, with numerous variants and a host of different names, is *mancala* – alleged to be one of the oldest games still to be played anywhere. Indeed, *mancala* tournaments now take place around the world.



▲ A Kenyan mancala board – players take turns to move their playing pieces from one pit to another, with the goal of capturing their opponent's pieces



Former collection of French botanist Jean-Baptiste Lamarck. Donated to the Museum of Toulouse in 2010.



Founded in 2003, the African Institute for Mathematical Sciences (AIMS) is a network of academic institutions in Cameroon, Ghana, Rwanda, Senegal, South Africa, and Tanzania. Its goal is to promote post-graduate education, research, and public engagement in the mathematical sciences in Africa.



One area of mathematics that has seen particular growth in African institutions in recent years is that of machine learning, driven by the foundation of such organisations as Data Science Africa and Deep Learning Indaba.

▼ The Rhind Mathematical Papyrus, the most complete and extensive surviving source on ancient Egyptian mathematics



© The Trustees of the British Museum

# World history of mathematics

## Mathematics in the Americas



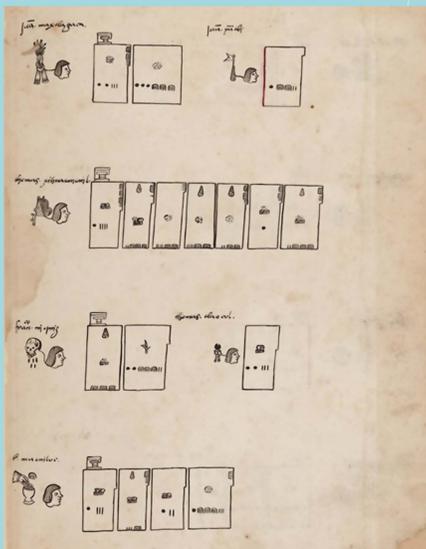
Mathematical Institute

The countries of North, Central, and South America feature prominently in the modern international mathematical community.

In the past, the various cultures of the continent, particularly of South America, developed a range of number systems, which were employed most extensively in connection with time-keeping.

### The Aztecs

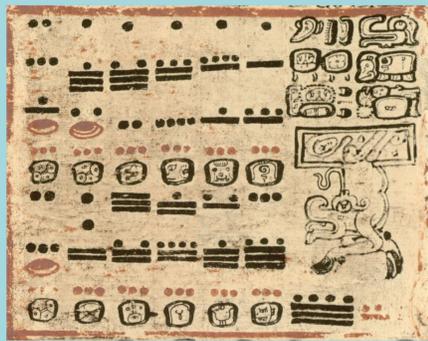
The use of base-20 number systems was widespread across South America, both linguistically and notationally. The Aztecs, for example, employed a non-positional base-20 number system with separate signs for 1, 20, 400, and 8000. These numerals may be found in surviving survey documents, which record areas and perimeters of fields – but we do not know how these were measured or calculated.



▲ An early 16th-century land survey, annotated in both Spanish and the Aztec language Nahuatl

### The Maya

The Maya culture emerged in Mesoamerica around 2,000 BCE, and elements of it survive to the present day. From around the fifth century BCE, the Maya employed an elaborate calendar, consisting of cycles within cycles. In the calendar, a 260-day count was combined with a year of 365 days; the two came back into step every 52 years, known as a *Calendar Round*. For dates on monuments, the *Long Count* was used; this was constructed around much larger units of time, some spanning centuries.



▲ Maya numerals in a rare surviving text of the 13th or 14th century CE

Dates in the long count were written using a base-20 positional numeral system, where a dot and a bar stood for '1' and '5', respectively; a picture of a shell often indicated an empty position.



▲ An inscription displaying a date in the Maya Long Count; numerals appear above the respective signs for units of time

### Quipu

Rather than recording information in a written form, some Andean civilisations, including the Incas of ancient Peru, developed a system of record-keeping using knots in bundles of wool and cotton cords called *quipu* ('talking knots').

The colours of the cords and the spacing of the knots were crucial to the encoding of information on the *quipu*. Although much of the content of the surviving *quipu* remains to be deciphered, we know that one of their major uses was the recording of numerical information.

### The Americas in the present day

During the past two centuries, the mathematical focus of the Americas has been on Canada and the United States. US-based mathematicians hold the largest number of Fields Medals; so far (2020) only one South American has won the award: Artur Ávila of Brazil.

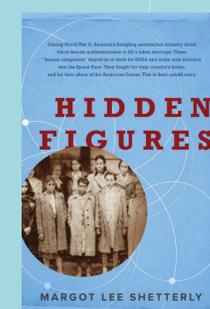


◀ Artur Ávila, awarded the Fields Medal in 2014 for 'contributions to dynamical systems theory, which have changed the face of the field, using the powerful idea of renormalization as a unifying principle'

However, the mathematical communities of other countries are now growing. In 1995, the Unión Matemática de América Latina y el Caribe was created, with the initial involvement of Argentina, Brazil, Chile, Colombia, Cuba, Mexico, Uruguay, Venezuela, and Peru; in 2018, Rio de Janeiro became the first city in the southern hemisphere to host an International Congress of Mathematicians. The study of the recent history of mathematics in the Americas is also changing: it has usually concentrated on white male mathematicians, but the contributions of other, previously-overlooked groups are now being recognised.



◀ Illustration from *El primer nueva corónica y buen gobierno* (*The First New Chronicle and Good Government*), 1615, by Peruvian Felipe Guaman Poma de Ayala; this is one of the few surviving accounts of the use of *quipu*



◀ The 2010 book *Hidden Figures: The American Dream and the Untold Story of the Black Women Who Helped Win the Space Race*



▲ An Incan *quipu* consisting of coloured and knotted cords, encoding numerical information

# World history of mathematics

## Mathematics in China



Mathematical Institute

Mathematics has a long pedigree in China, both as an accomplishment expected of Confucian literati, and as a subject that was key to training as a civil servant.

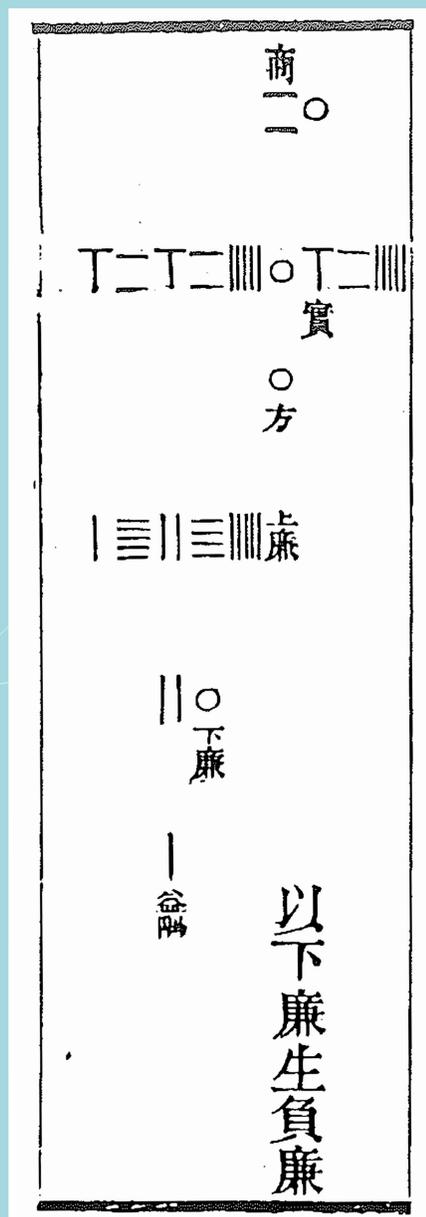
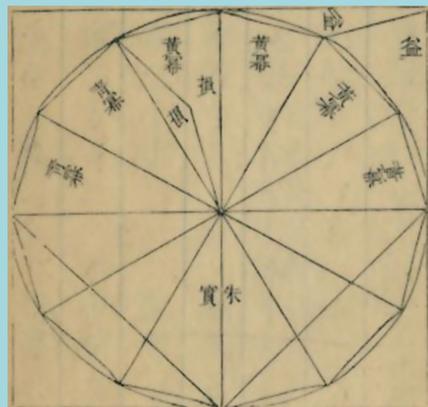
### The Nine Chapters on the Mathematical Art

One of the most influential of the traditional Chinese mathematical texts was the *Jiǔzhāng suànshù* 九章算術 (*The Nine Chapters on the Mathematical Art*), probably dating originally from c. 300 BCE, but in use for over 1,000 years as a manual for the training of administrators. The *Jiǔzhāng suànshù* consists of 246 arithmetical problems and their methods of solution. Its seventh and eighth chapters, for example, deal with the solution of systems of up to five simultaneous equations in five unknowns. The method used, *fāngchéng* 方程, is equivalent to the process known in the West as Gaussian elimination, and may have been influenced by the use of counting boards in China.



▲ The *Jiǔzhāng suànshù* was transmitted down the centuries via commentaries that often added substantially to its content. A particularly famous commentary was that supplied by Liu Hui 劉徽, perhaps in the third century CE

▼ Liu Hui's method for the calculation of the area of a circle, given its diameter and circumference implicitly takes 3.14024 as the value of  $\pi$



▲ The handwritten numerals shown above derive from Chinese counting rods: bamboo rods were used to represent numbers within spaces on a grid, with red rods for positive numbers, and black for negative

### Master Sun's Mathematical Manual

A later text that also became required reading for civil servants was the *Sūnzǐ suànjīng* 孫子算經 (*Master Sun's Mathematical Manual*), probably written in the third century CE.

Motivated by calendrical problems, the *Sūnzǐ suànjīng* deals with congruences as well as ordinary arithmetic, and is the earliest source for what we now call the Chinese Remainder Theorem on the solution of simultaneous congruences.



孫子算經卷上  
唐劉徽注  
度之所起起於忽欲知其忽蓋吐絲為忽十忽為一絲十絲為一毫十毫為一釐十釐為一分十分為一寸十寸為一尺十尺為一丈十丈為一引五十尺為一端四十尺為一疋六尺為一步二百四十步為一畝三百步為一里稱之所起起於黍十黍為一粟十粟為一銖二十四銖為一兩十六兩為一斤三十斤為一鈞

▲ Page from an edition of the *Sūnzǐ suànjīng* printed more than 1,000 years after the original text was written; beyond his name, we know nothing about the alleged author, Sun Zi

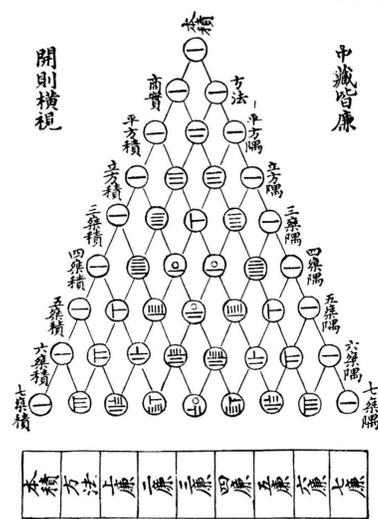
### Wang Zhenyi

Although mathematical learning in China was confined largely to men, some women did also engage in scientific studies. Most notable was Wang Zhenyi 王貞儀 (1768–97), who wrote treatises on both astronomy (the procession of the equinoxes) and mathematics (trigonometry).

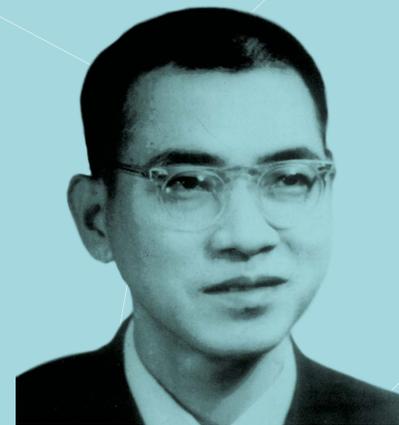
Because traditional Chinese mathematics focused on methods rather than proofs, it has often been compared unfavourably with the ancient Greek mathematics that has traditionally provided the benchmark for how mathematics 'should' be done. Nowadays, however, historians of mathematics strive to treat ancient Chinese mathematics, and other non-Western mathematical traditions, purely on their own terms.

China occupies a prominent position in international mathematics; in August 2002, Beijing became only the second Asian city to host the International Congress of Mathematicians.

### 古法七乘方圖



▲ The Yang Hui triangle 杨辉三角形 (a.k.a. Pascal's triangle), as found in the *Sīyuán yùjiàn* 四元玉鑑 (*Jade Mirror of the Four Unknowns*, 1303) by Zhū Shìjié 朱世傑 (1249–1314)



▲ Chen Jingrun 陳景潤 (1933–96), who made progress towards the proof of the Twin Prime Conjecture by proving in 1966 that there are infinitely many primes  $p$  for which  $p + 2$  is either prime or a product of two primes

▼ Wang Zhenyi 王貞儀 (1768–97), astronomer and mathematician



# World history of mathematics

## Mathematics in India



Mathematical Institute

India has a mathematical tradition that goes back many thousands of years.

The Indus Valley Civilisation, which emerged around 3,000 BCE, had a centralised system of weights and measures, along with measuring instruments. Although the Indus script has not yet been deciphered, India nevertheless boasts mathematical texts whose originals date from 800–500 BCE (but are now known only from later copies) in the form of the *Sūlbāsūtras* शूल्बसूत्रम्, which concern geometry and the construction of sacrificial altars.



▲ The Bakhshali manuscript is one of the oldest surviving original Indian mathematical texts (estimates place it anywhere between the second and the tenth centuries CE). It is a compendium of mathematical problems and rules for their solution. Depending on its date, it may contain the oldest known example of the symbol 0 for zero.



▲ Statue of Āryabhaṭa outside the Inter-University Centre for Astronomy and Astrophysics in Pune, Maharashtra, India

### Classical Period

The so-called ‘classical period’ of Indian mathematics began in the middle of the first millennium CE with the work of Āryabhaṭa I आर्यभट्ट (476–550 CE). His Sanskrit treatise *Āryabhaṭīya* आर्यभटीयम् (c. 510 CE)

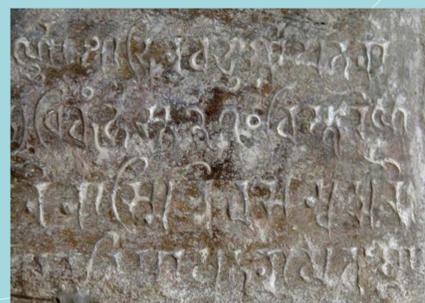


▲ The *Tantrasaṅgraha* तंत्रसंग्रह (1501), an astronomical treatise by Nīlakaṇṭha Somaṃjī नीलकण्ठ सोमयजी (1444–1544), containing power series expansions for trigonometric functions, written on palm leaves in Sanskrit verse

deals mostly with plane and spherical trigonometry, but also covers the solution of various types of equations, and contains an approximation of  $\pi$  as 3.1416. Other figures from the classical period are Brahmagupta ब्रह्मगुप्त (c. 598–668 CE), who was the first scholar to set out rules for performing arithmetic with negative numbers and zero, and Bhāskara I भास्कर प्रथम (c. 600–680 CE), author of three astronomical treatises.

### Kerala School

During the 14th–16th centuries CE, a school of mathematics and astronomy flourished in Kerala in southern India. Often motivated by astronomical problems, the school produced treatises on arithmetic, algebra, geometry, approximation of roots of equations, and magic squares, and also developed power series expansions for trigonometric functions, two centuries before these same ideas appeared in Europe.



▲ The oldest surviving undisputed occurrence of the symbol 0 for zero appears in an inscription in Chaturbhuj Temple in Gwalior, Madhya Pradesh, India

### Numerals

Indian scholars employed a decimal positional number system from early in the first millennium CE. Initially, empty columns were indicated with a word meaning ‘emptiness’, but eventually a dot, and then a circle, came to be used. This system of numerals was adopted by Middle Eastern scholars in the ninth century, and was transmitted to Europe in the following centuries – both by Italian merchants and via Islamic Iberia – to become the number system that we use today, often termed ‘Hindu-Arabic numerals’.

▼ Participants of the 2012 IWM Annual Conference



### Ramanujan

India’s most famous mathematician of the twentieth century was Srinivasa Ramanujan சீனிவாச இராமானுசன் (1887–1920). Born in present-day Tamil Nadu, Ramanujan was largely self-taught in mathematics.

In 1913, he began a correspondence with G. H. Hardy in Cambridge, which led to his being invited there the following year. Ramanujan remained in Cambridge until shortly before his early death in 1920, aged just 32; in 1918, he was elected both a Fellow of the Royal Society and of Trinity College, Cambridge.



▲ Srinivasa Ramanujan, who made profound contributions to analysis and number theory; with Hardy, he obtained an asymptotic formula for the partition function

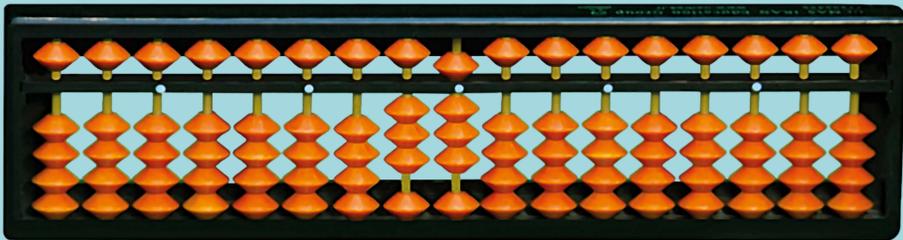
### Indian Women and Mathematics

Like many countries, India has an association devoted to promoting the involvement of women in mathematics: Indian Women and Mathematics, founded in 2009. It holds annual conferences and regional workshops throughout India.

# World history of mathematics

## Mathematics in Japan

During the seventeenth to nineteenth centuries, Japanese mathematicians developed a unique form of mathematics, known as *wasan*.



▲ A *soroban*, the Japanese abacus

After Japan opened up to the West in the mid-nineteenth century, its government decreed that Western mathematical techniques (洋算 *yōsan*) should be adopted. The exception to this was that children should still be taught to use the Japanese abacus, the *soroban* 算盤.

In earlier centuries, Japanese mathematics had been heavily influenced by texts from China, and European missionaries had taught Western mathematics in Japan in the early seventeenth century, but during Japan's period of isolation from the rest of the world, from the mid-seventeenth century onwards, a Japanese form of mathematics had flourished: *wasan* 和算.

In the 17th century there arose in Japan a tradition of posing ever harder and more intricate, and often rather artificial, mathematical problems, often taken from Chinese textbooks, whose goal was to challenge the reader rather than to develop a comprehensive understanding of the underlying mathematics. It was out of such

▼ Page from a mid-seventeenth century arithmetic textbook *Shinpen Jinkoki* 新編塵劫記, compiled by Yoshida Mitsuyoshi 吉田光由 (1598–1673). The problem illustrated is equivalent to the so-called 'Josephus problem' that was posed in Mediaeval Europe



problems, and in reaction to them, that the more systematic *wasan* gradually developed.

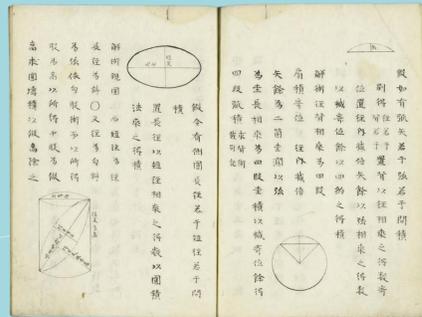
### Seki Takakazu, the Japanese Newton

Seki Takakazu 関孝和 (?–1708) is considered the father of *wasan*. Moving away from a view of mathematics as a collection of isolated problems, he sought general methods. In contrast to Western mathematics, *wasan* was *inductive* rather than *deductive*: mathematical enthusiasts (Seki himself was an accountant by day) carried out arithmetical experimentations to establish patterns and results.



▲ Seki Takakazu 関孝和, sometimes referred to as 'the Japanese Newton'

Using such an approach, Seki was one of the first mathematicians to discover the Bernoulli numbers, and to develop the idea of a determinant. He is also credited with the foundation of the branch of mathematics known as *enri* 円理, or *circle principles*, which sought general methods for calculating areas, volumes, and lengths of curves, and therefore paralleled the development of calculus in Europe.



▲ A page from Seki's textbook *Kaikendai no hō* 解見題之法 (*Methods of Solving Explicit Problems*), featuring techniques for calculating areas of plane figures and volumes of some solids

### Sangaku

Alongside the more systematic *wasan*, the tradition of posing isolated problems remained strong, driven by mathematical hobbyists. Rival mathematical sects developed across Japan, with little interaction between different groups. These schools often had religious links, and displayed their mathematical achievements via votive tablets, *sangaku* 算額, hung outside Shinto shrines or Buddhist temples. These showed arithmetical problems or geometrical figures with numerical solutions, but never indicated how the problem had been solved.



▲ Example of a *sangaku* (c.1846) from Ito Shrine in Osaka Prefecture  
◀ *Sangaku* (1861) from Sōzume shrine in Okayama City, showing women and children performing calculations on *soroban*



During the early 20th century, many Japanese students travelled to Europe – Germany, in particular – to study mathematics; in the second half of the century, however, the focus shifted to the USA and to Japanese institutions. In 1990, Kyoto became the first Asian city to host an International Congress of Mathematics.



▲ To date (2020), Japan boasts three Fields Medalists: (clockwise from above) Kodaira Kunihiko 小平邦彦 (1915–97), Hironaka Heisuke 広中平祐 (b. 1931), and Mori Shigefumi 森重文 (b. 1951)



# World history of mathematics

## Mathematics in the Middle East

Some of the world's earliest mathematics was developed in the Middle East by the Sumerian and Babylonian cultures after c. 3,000 BCE, much of which was eventually transmitted to ancient Greece.

A later Middle Eastern mathematical and astronomical school flourished under Islamic influence, particularly during the 9th and 10th centuries CE. The pursuit of knowledge was seen as having religious merit, and certain verses in the *Quran* were interpreted as encouraging scientific enquiry.

### Babylonian mathematics

The scribes of ancient Mesopotamia employed a base-60 positional numerical notation, many examples of which appear on surviving clay tablets. Using this notation, the scribes were able to solve a range of arithmetical and geometrical problems, as well as to develop a sophisticated astronomy – their use of base-60 is preserved in our division of an hour in 60 minutes, a minute into 60 seconds, and a circle into 360 degrees.



▲ Mathematical clay tablet, dating from 1900–1600 BCE, showing an area calculation

### House of Wisdom

The Bayt al-Hikmah بيت الحكمة (House of Wisdom) was founded in Baghdad at the end of the eighth century CE. It was created to produce Arabic translations of ancient Greek texts, but soon became a centre for original scholarship as well. Developments were made in particular in both mathematics (especially algebra) and astronomy (especially spherical trigonometry).

### Al-Khwārizmī

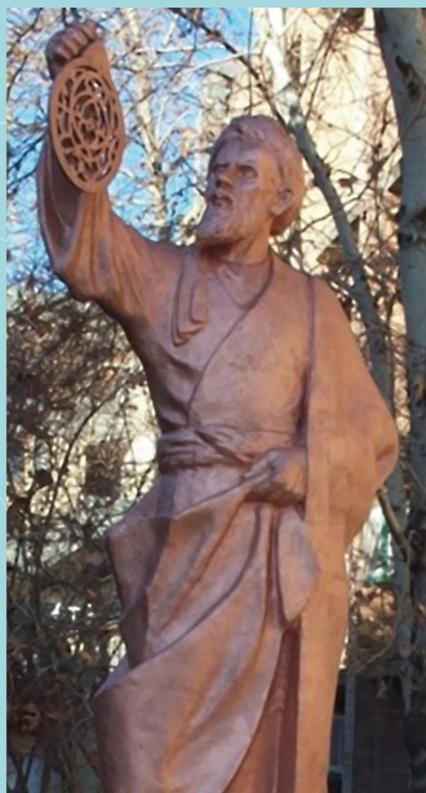
One of the Bayt al-Hikmah's most famous figures during the ninth century CE was the Persian scholar Muḥammad ibn Mūsā al-Khwārizmī محمد بن موسى خوارزمي (c. 780–850 CE).

▼ 13th-century depiction of scholars in the Abbasid Library at the Bayt al-Hikmah



In his *Compendious Book on Calculation by Completion and Balancing* الكتاب المختصر في حساب الجبر والمقابلة (c. 820 CE), al-Khwārizmī set out methods for the systematic solution of quadratic equations, which for him fell into six distinct cases because he did not use negative numbers or zero. The 'al-jabr' ('balancing') of the text's Arabic title was taken over into European languages as our word 'algebra'. Al-Khwārizmī's rules for

▼ Statue of al-Khwārizmī that stands in front of the Faculty of Mathematics of Amirkabir University of Technology in Tehran



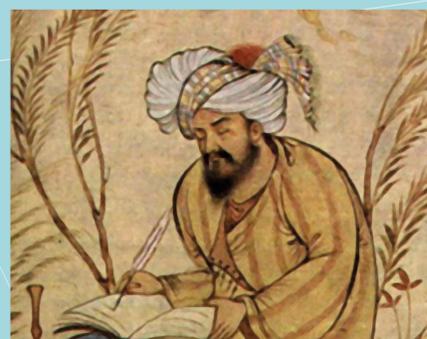
solving equations were presented as step-by-step procedures; a Latinised version of his name gives us the word 'algorithm'.



▲ Page from al-Khwārizmī's *Compendious Book* giving a geometrical justification for his rule for the solution of a quadratic equation of the form  $x^2 + bx = c$  (for  $b$  and  $c$  both positive)

### Omar Khayyam

Two hundred years later, al-Khwārizmī's methods for solving quadratic equations were extended to the case of cubics by Omar Khayyam عمر خیام (1048–1131), who lived in modern-day Iran. Famous in the West as a poet, Khayyam also studied astronomy and mathematics. As an astronomer, he devised a new solar calendar, while his contributions to mathematics covered both geometry and algebra. In particular, he provided solutions of cubic equations, on a case-by-case basis, as intersections of conic sections.



▲ Omar Khayyam, Persian mathematician, astronomer, and poet

▼ Omar Khayyam's solution of a cubic equation of the form  $x^3 + cx + d = bx^2$  (for  $b, c, d$  positive)



As well as producing new ideas, Mediaeval Islamic scholars also helped to preserve the knowledge of the ancient world, via their systematic translation of writings whose originals are now lost. Certain texts, such as parts of Diophantus' *Arithmetica*, a Greek compendium of algebraic problems from the third century CE, are known only through surviving Arabic translations.

Maryam Mirzakhani مریم میرزاخانی (1977–2017) was born in Iran, but subsequently moved to the United States. In 2014, she became the first woman to win a Fields Medal for "her outstanding contributions to the dynamics and geometry of Riemann surfaces and their moduli spaces."



▲ Maryam Mirzakhani (1977–2017), the first woman to win a Fields Medal

# World history of mathematics

## Mathematics in Oceania



Mathematical Institute

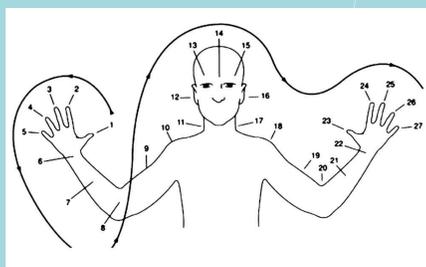
The various cultures of Oceania provide us with many examples of mathematics appearing in unexpected places.

The same mathematical ideas have often appeared implicitly in different places around the globe, but tend to be articulated and conceptualised in distinct ways. As scholars, we must therefore be flexible in what we interpret as ‘mathematics’.

We can analyse the latent mathematical notions of different cultures using Western ideas, but this doesn’t necessarily mean that an awareness of these same concepts in their Western formulation is present in the original ideas: an appreciation of symmetry, for example, does not automatically signal a knowledge of the notion of a group.

### Number systems

The many cultures and languages of Oceania provide a variety of examples of different counting systems, constructed on a range of bases, including 2, 4, 5, 6, and 10. The Oksapmin of Papua New Guinea employ a base-27 counting system which involves the enumeration of different parts of the body.

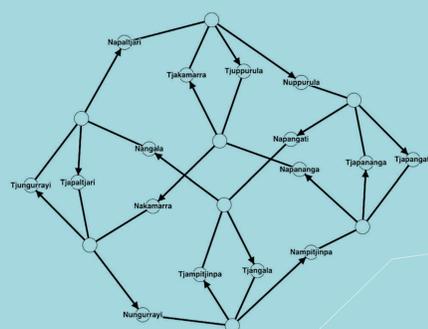


▲ The enumeration of parts of the body, according to the Oksapmin base-27 counting system

### The logic of kinship

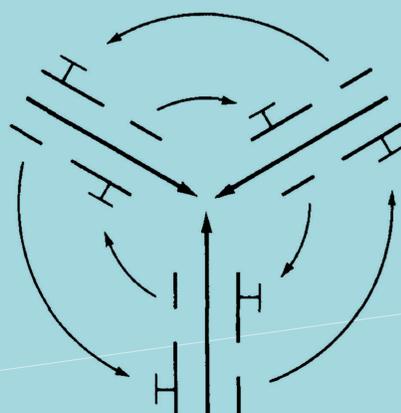
Systems of kinship have particular importance amongst many Australian Aboriginal peoples. Such systems, whereby each person is assigned to a traditional subgroup of the population, encode the rights and obligations of individuals within society, and determine who may marry whom.

A kinship system that has received a lot of attention is that of the Warlpiri people of Australia’s Northern Territory. The entire



▲ A graph showing the interrelations between the sections of Warlpiri society; unlabelled nodes indicate permissible marriages, and arrows indicate the subsection to which a child of that marriage will belong

Warlpiri population is divided into eight named subsections, and the subsections to which parents belong determine those of their children in a systematic way. If we label these interrelationships in a certain way, we find the structure of the dihedral group of order 8.



▲ The drawing of continuous figures in sand has an important ritualistic purpose for the Malekula people of Vanuatu. In particular, diagrams such as that pictured here were used to illustrate Malekula kinship relationships for early Western visitors. With appropriate labelling, we may identify the structure of a dihedral group of order 6

### Mū tōrere

Games of strategy are played in many cultures around the world, and the Māori of New Zealand are no exception. The game of *mū tōrere* is played with counters on a grid shaped like an eight-pointed star. Players take it in turns to move their counters to adjacent empty



▲ Each *mū tōrere* player begins with four counters each, arranged around the outer points of the playing grid. There are 86 attainable configurations on the game board, including 3 winning positions for each player

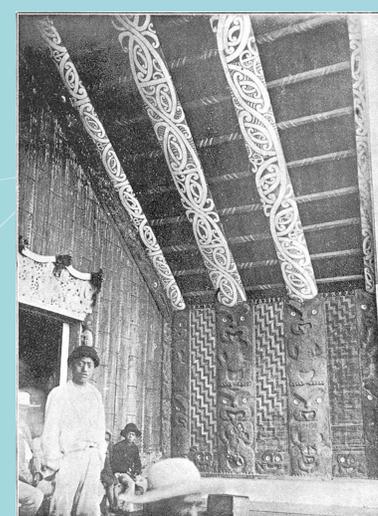
points with a view to blocking their opponent. A combinatorial analysis of versions of the game played on grids having more than or fewer than eight points suggests that the eight-pointed version provides just the right level of complexity to make for an interesting game.



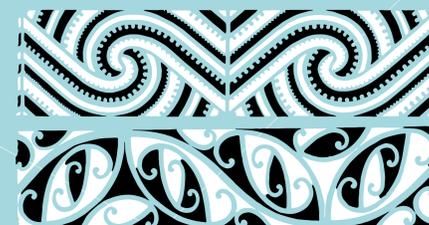
▲ A specially made *mū tōrere* board. The game is equally well played with pebbles on a grid marked out in sand, for example

### Māori rafter patterns

*Kōwhaiwhai* are traditional painted motifs that appear on the rafters of Māori meeting houses. Created by a master carpenter, these are geometric patterns that tell stories about the people who built the meeting house, represent ancestors, heroes or gods, or reflect aspects of the natural environment. Notions of symmetry and balance are often central to these designs, which lend themselves to analysis via Western mathematical techniques.



▲ *Kōwhaiwhai* in a house of the Ngāti Porou at Waiomatatini, East Cape, New Zealand



▲ There is a formal structure to *kōwhaiwhai*, whose carving must be both ritually and technically correct



In 2018, the University of Sydney launched its global Mathematical Research Institute, with the goals of drawing leading mathematical scientists from around the world to work with Australian collaborators, and of increasing public engagement with mathematics.

▼ The University of Sydney Main Quadrangle

